International Journal of Theoretical Physics, Vol. 37, No. 6, 1998

Potentials with Convergent Schwinger–DeWitt Expansion

V. A. Slobodenyuk[1]

Received November 15, 1997

Convergence of the Schwinger–DeWitt expansion for the evolution operator kernel for special class of potentials is studied. It is shown that this expansion, which is in the general case asymptotic, converges for the potentials considered (widely used, in particular, in one-dimensional many-body problems), and that convergence takes place only for definite discrete values of the coupling constant. For other values of the charge, a divergent expansion determines the kernels having essential singularity at the origin (beyond the usual δ-function). If one considers only this class of potentials, then one can avoid many problems connected with asymptotic expansions, and one gets a theory with discrete values of the coupling constant that is in correspondence with the discreteness of charge in nature. This approach can be applied to quantum field theory.

1. INTRODUCTION

In quantum theory, expansions in different parameters such as the coupling constant (Bender and Wu, 1969, 1971a, b, Lipatov, 1977), the WKB expansion, the short-time Schwinger–DeWitt expansion (Schwinger, 1951; DeWitt, 1965; 1975), the perturbation expansion in phase space (Barvinsky *et al.*, 1995), the $1/n$ expansion (Popov *et al.*, 1992), etc., are, as a rule, asymptotic. This circumstance imposes essential restrictions on their use, makes the theory incomplete, and compels one to look for ways of overcoming these restrictions, either by summation of divergent series with special methods (see, e.g., Kazakov and Shirkov, 1980), or by constructing new convergent expansions (Halliday and Suranyi, 1980; Ushveridze, 1983; Sissakian and Solovtsov, 1992), or by creating different approximate methods taking into consideration the so-called nonperturbative effects.

[1] Physical-Technical Department, Ulyanovsk State University, 432700 Ulyanovsk, Russian Federation; e-mail: slob@themp.univ.simbirsk.su.

0020-7748/98/0600-1753$15.00/0

Interesting information about a possible way of overcoming the problem of divergences in some cases may be obtained from investigation of the time dependence of the evolution operator kernel with the help of the Schwinger–DeWitt expansion. For example, Osborn and Fujiwara (1983) studied a class of potentials given by the family of bounded and continuous functions that are formed from the Fourier transforms of complex bounded measures. An important feature of this expansion is that after factorization of the contribution of the free kernel ("free" case corresponds to $V \equiv 0$), having at $t = 0$ a singularity in the form of a δ-function in space variables, one can concentrate on the remaining part (denote it as F) which, according to the initial condition, should be equal to 1 when $t = 0$. To understand the behavior at $t = 0$ it is necessary to make the analytical continuation, into the complex plane t. When this continuation is made for the kernel, then its analytical properties are masked by the singularity, which provides δ-like behavior in space variables. But if the factorization of the free part of the kernel is made, then the rest of the function can be continued into the entire complex t plane and one can accurately examine its properties in the neighborhood of the origin.

If the Schwinger–DeWitt expansion is convergent, then the point $t = 0$ is regular and the initial condition is fulfilled in the rigorous sense. But if this expansion is divergent [note that usually it is treated as asymptotic (Osborn and Fujiwara, 1983; Slobodenyuk 1995, 1996a, b)], then the point $t = 0$ is essentially a singular point for the function F. In this case the initial condition may be fulfilled only in the asymptotic sense. The function F tends to 1 when $t \to 0$ along the real positive semiaxis as a continuous function, but it is not analytic at $t = 0$ and it does not have any meaning at this point.

Nevertheless, it is enough to fulfill the initial condition even in the asymptotic sense that unambiguous solution of the evolution problem exists. So, divergence of the Schwinger–DeWitt expansion does not put any formal restrictions on the choice of the potentials in the quantum theory. But using potentials with a divergent expansion (if the exact solution is not known) is usually connected with problems of different divergences (see, e.g., Section 4).

There exists the possibility to avoid many of these problems in some cases. If one considers the potentials for which the Schwinger–DeWitt expansion converges, then one may get convergent representations for the kernel and other physical values. Such nontrivial potentials really exist. This paper is devoted to consideration of some examples of such potentials and to proving convergence of the expansion for them. These are the potentials being widely used in one-dimensional many-body problems (Olshanetsky and Perelomov, 1983; Calogero *et al.*, 1975; Sutherland, 1971, 1972). For definite discrete values of the coupling constant their expansions are convergent in the entire complex plane t. For other values of the charge the expansions are asymptotic. The existence of such potentials is very interesting.

Moreover, convergence of the expansion only for discrete values of the coupling constant may be connected with the discreteness of charge in nature.

2. METHOD

The evolution operator kernel of the Schrodinger equation in the one-dimensional case is the solution of the problem

$$i\frac{\partial}{\partial t}\langle q', t|q, 0\rangle = -\frac{1}{2}\frac{\partial^2}{\partial q'^2}\langle q', t|q, 0\rangle + V(q')\langle q', t|q, 0\rangle \tag{1}$$

$$\langle q', t = 0|q, 0\rangle = \delta(q' - q) \tag{2}$$

Here and everywhere below, dimensionless values, which are derived from dimensional ones in an obvious way, are used for the sake of convenience. The variable t is treated as a complex one. If one means the proper Schrodinger equation, then t is real. We imply that $V(q)$ does not apparently depend on time.

As is well known, in the free case ($V \equiv 0$) the solution of the problem (1), (2) is

$$\langle q', t|q, 0\rangle = \frac{1}{\sqrt{2\pi t}}\exp\left\{i\frac{(q' - q)^2}{2t}\right\} \equiv \phi(t; q', q) \tag{3}$$

The function ϕ has an essential singularity at $t = 0$, but this singularity is such that the initial condition (2) is fulfilled.

When interaction is present the kernel can be represented as

$$\langle q', t|q, 0\rangle = \frac{1}{\sqrt{2\pi i t}}\exp\left\{i\frac{(q' - q)^2}{2t}\right\} F(t; q', q) \tag{4}$$

and one can write for F the expansion (the short-time Schwinger–DeWitt expansion)

$$F(t; q', q) = \sum_{n=0}^{\infty} (it)^n a_n(q', q) \tag{5}$$

which, as a rule, is asymptotic and is usually utilized only in that sense. We shall use the representation (4), (5) to test the analytic properties of the evolution operator kernel in variable t, and, in particular, ascertain its behavior for $t \to 0$.

For this purpose let us derive from (1) the equation for F

$$i\frac{\partial F}{\partial t} = -\frac{1}{2}\frac{\partial^2 F}{\partial q'^2} + \frac{q' - q}{it}\frac{\partial F}{\partial q'} + V(q')F \tag{6}$$

Because the factorized function ϕ still satisfies the initial condition (2), then F should satisfy the initial condition

$$F(t = 0; q', q) = 1 \tag{7}$$

It seems at first sight that it is possible to add to the right-hand side of (7) an arbitrary function of $q' - q$ which vanishes at $q' = q$. However, this is not true. The equation for the coefficient a_0

$$(q' - q) \frac{\partial a_0(q', q)}{\partial q'} = 0$$

taken from general recursion relations for $a_n(q', q)$, with the condition $a_0(q, q) = 1$, determines unambiguously

$$a_0(q', q) = F(0; q', q) = 1$$

Our problem (1), (2) has a physical sense only for real, positive τ, where $\tau = it$ (if the heat equation and heat kernel are considered), or for real t (if the quantum mechanical evolution equation is considered). The same restrictions initially hold for equation (6), too. But we can analytically continue the function F into the complex plane of the variable t using the differential equation (6) with condition (7). Now the variable t may vary in the entire complex plane t. There is no restriction Re $t > 0$ as holds for the analytic semigroup.

If q is a regular point of the function $V(q)$ and at any domain the expansion in powers of $\Delta q = q' - q$ hold.

$$V(q') = \sum_{k=0}^{\infty} \Delta q^k \frac{V^{(k)}(q)}{k!} \tag{8}$$

(the notation

$$V^{(k)}(q) \equiv \frac{d^k V(q)}{dq^k}$$

is used here and will be used below), then one can use the concrete form of the coordinate dependence of the coefficients a_n

$$F(t; q', q) = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (it)^n \Delta q^k b_{nk}(q) \tag{9}$$

It is obvious that

$$\sum_{k=0}^{\infty} \Delta q^k b_{nk}(q) = a_n(q', q) = -Y_n(q', q)$$

where Y_n are the functions introduced in Slobodenyuk (1993). The behavior of Y_n was studied in Slobodenyuk (1995), using the representation adduced in Slobodenyuk (1993).

Substitution of (9), (8) into (6) leads to recurrence relations for the coefficients b_{nk}

$$b_{nk} = \frac{1}{n + k}\left[\frac{(k+1)(k+2)}{2} b_{n-1,k+2} = \sum_{m=0}^{k} \frac{V^{(m)}(q)}{m!} b_{n-1,k-m}\right] \tag{10}$$

with condition $b_{0k} = \delta_{k0}$. Specifically,

$$b_{1k} = -\frac{V^{(k)}(q)}{(k+1)!} \tag{11}$$

Expressions (9), (10) determine a formal solution of problem (6), (7). As to the expansion in powers of Δq in (9), one may expect that its convergence range is equal to one for expansion (8) of the potential. The series in t in (9) is usually treated as divergent. At first sight, it is always so. Let us estimate the convergence of the series in (9).

First let n be fixed and $k \to \infty$. Expressing $b_{n-1,k+2}$ from (10) via the coefficients with smaller k, we come to some linear combination of coefficients of type $b_{n_0,0}$ and $b_{n_1,1}$ with any indexes n_0, n_1 (for the sake of brevity we shall write further only the terms with $b_{n_0,0}$, implying that the same statements hold for the terms with $b_{n_1,1}$). The main growth for large k takes place if the second index of b_{nk} is diminished; (a) using the term $V^{(k)} b_{n-1,0}/k!$, (b) using the expression on the left-hand side of (10).

In case (a) we get for $k \to \infty$

$$|b^{(a)}_{n-1,k+2}| \sim \frac{2}{(k+1)(k+2)} \frac{|V^{(k)}|}{k!} |b_{n-1,0}| \tag{12}$$

Because the series (8) converges at some circle with radius $R(q)$ the estimate

$$\frac{|V^{(k)}|}{k!} \sim \frac{1}{R^k(q)}$$

holds for $k \to \infty$. So, for every fixed n and for $k \to \infty$ we have

$$|b^{(a)}_{nk}| \sim \frac{|b_{n0}|}{R^k(q)} \tag{13}$$

Contributions of type (13) correspond to the expansion in Δq, which is convergent for every fixed n with convergence range $R(q)$.

In the case (b) for $k \to \infty$ we get

$$|b^{(b)}_{n-1,k+2}| \sim \frac{2^{k/2+1}(n+k)!}{k!\,(n+k/2-1)!} |b_{n+k/2,0}|$$

The behavior of $b^{(b)}_{nk}$ for $k \to \infty$ depends on the behavior of b_{n0} for $n \to \infty$.

If b_{n0} decreases when $n \to \infty$ or increases more slowly than $\Gamma(\alpha n)$ (α is any positive number), then

$$|b_{nk}^{(b)}| \sim \frac{|b_{n0}|}{\Gamma(k/2)}$$

for $k \to \infty$, i.e., these contributions will disappear at large k. If b_{n0} increases as $\Gamma(\alpha n)$ [here $0 < \alpha \leq 1$; Slobodenyuk (1995) showed that α cannot be larger than 1], then for $k \to \infty$ and $\alpha < 1$

$$|b_{nk}^{(b)}| \sim \frac{|b_{n0}|}{\Gamma\frac{1}{2}\,(1 - \alpha)k}$$

so these contributions will disappear, too, with the growth of k. If $\alpha = 1$, then the following estimate will hold (n is fixed, $k \to \infty$)

$$|b_{nk}^{(b)}| \sim |b_{n0}| k^c \rho^k \tag{14}$$

In this case the expansion in Δq in (9) will have a finite convergence range, too, but it will be equal to the minimum of the two values $R(q)$ and ρ.

Now let us examine the behavior of $|b_{n0}|$ (the same will be also correct for $|b_{n1}|$) when $n \to \infty$. Consider the decrease of n to 1 by means of the first term on the right-hand side of (10),

$$|b_{n0}| \sim \frac{|b_{n-1,2}|}{n} \sim \cdots \sim \frac{(n-1)!}{2^{n-1}} |b_{1,2n-2}| = \frac{(n-1)!}{2^{n-1}} \frac{|V^{(2n-2)}|}{(2n-1)!} \tag{15}$$

Because $|V^{(k)}| \sim k!/R^k(q)$ for $k \to \infty$, then for $n \to \infty$ we get

$$|b_{n0}| \sim \frac{(n-1)!}{2^{n-1}(2n-1)} \sim n! \tag{16}$$

The contributions taken into account in (15) provide the main growth only for the potentials for which $R(q) < \infty$. If potentials with $R(q) = \infty$ are considered (e.g., polynomial ones), then one might conclude from (15) that $|b_{n0}| \sim 1/n!$. But this is not so, in fact. As it shown in Slobodenyuk (1995), the combination of contributions of the first term and terms from the sum over m in (10) leads to an estimate of type $|b_{n0}| \sim \Gamma(\alpha n)$.

So, for arbitrary potentials the series in t in (9) is divergent. But in our estimates, in fact, absolute values of all contributions to every coefficient b_{nk} were summed. Nevertheless, for some potentials the cancellation of different terms may occur. It can lead to convergence of the expansion in (9). For the potentials considered in Sections 3–5 this cancellation takes place only for definite values of the coupling constant.

Note that we really test expansion (9) for absolute convergence. So it is enough for the convergence of double series that (9), in which, instead of

b_{nk} we take absolute values $|b_{nk}|$, would converge for any order of summation. Our consideration corresponds to the following order: at first the series over k for every fixed n are summed and then summation over n is made. If one assumes that there is convergence of the series in the index n, then, as shown before, the convergence in index k will take place at every fixed n, and to establish the convergence of the series in index n it is enough to determine the behavior of the coefficients b_{n0}, b_{n1} only (but not all b_{nk}) at $n \to \infty$.

3. MODIFIED PÖSCHL–TELLER POTENTIAL

Let us introduce standard notation for the coupling constant $g = \lambda(\lambda - 1)/2$ ($\lambda > 0$) and investigate the modified Poschl–Teller potential

$$V(q) = -\frac{\lambda(\lambda - 1)}{2} \frac{\beta^2}{\cosh^2(\beta q)} \tag{17}$$

for the convergence of the expansion (9).

Because the constant β is connected with the choice of length scale, one can put $\beta = 1$ without the restriction of generality. Further, for the sake of brevity we shall denote

$$f(q) = -\frac{1}{\cosh^2 q} \tag{18}$$

Then the potential reads $V(q) = gf(q)$.

The potential (17) has the expansion of type (8) about every real point q. Its convergence range is equal to $R(q) = [(\pi/2)^2 + q^2]^{1/2}$ and is determined by the distance to the nearest singularities of the function $1/\cosh^2 q$ placed at the points $q = \pm i\pi/2$. The derivatives can be calculated as follows:

$$V^{(k)}(q) = g f^{(k)}(q) \tag{19}$$

where $f^{(k)}$ are represented as expansions in powers of f,

$$f^{(2n)}(q) = \sum_{l=1}^{n+1} a_l^{(2n)} f^l(q) \tag{20}$$

$$f^{(2n+1)}(q) = \sum_{l=1}^{n+1} l a_l^{(2n)} f^{l-1} f^{(1)} = \sum_{l=1}^{n+1} a_l^{(2n+1)} f^{l-1} f^{(1)} \tag{21}$$

To obtain all coefficients $a_l^{(k)}$, it is enough to put $a_l^{(0)} = \delta_{l1}$ and take into account

$$(f^{(1)})^2 = 4f^3 + 4f^2$$

For $a_l^{(2n)}$ one has the recursion relations

$$a_l^{(2n)} = 4l^2 a_l^{(2n-2)} + 4(l-1)(l-1/2) a_{l-1}^{(2n-2)} \tag{22}$$

So, every derivative of the function $f(q)$ is represented as a polynomial in powers of this function.

From (10) one gets for the potential (17)

$$b_{nk} = \frac{1}{n+k}\left[\frac{(k+1)(k+2)}{2} b_{n-1,k+2} - \frac{\lambda(\lambda-1)}{2}\sum_{m=0}^{k}\frac{f^{(m)}}{m!} b_{n-1,k-m}\right] \tag{23}$$

where the derivatives $f^{(m)}$ are calculated via (20)–(22).

According to the note at the end of Section 2, it is enough for testing the convergence of series (9) to examine the behavior at $n \to \infty$ of the coefficients b_{n0}, b_{n1} only. Introduce in this connection the functions

$$B_k(t, q) = \sum_{n=0}^{\infty} t^n b_{nk}(q) \tag{24}$$

and consider them for $k = 0, 1$.

The analysis of relations (23) taking into account (20)–(22) shows that B_0, B_1 can be represented in the form

$$\begin{aligned} B_0(t, q) &= 1 + \sum_{n=1}^{\infty}(it)^n \sum_{l=1}^{n}\frac{(-1)^l}{l!} f^l(q)\, \beta_{nl} \times \prod_{j=1}^{l}\left(\frac{\lambda(\lambda-1)}{2} - \frac{j(j-1)}{2}\right) \\ &= 1 + \sum_{n=1}^{\infty}(it)^n \sum_{l=1}^{n}\frac{(-1)^l}{l!} f^l(q)\beta_{nl}\frac{\Gamma(\lambda+l)}{2^l\Gamma(\lambda-l)} \end{aligned} \tag{25}$$

$$\begin{aligned} B_1(t, q) &= \sum_{n=1}^{\infty}(it)^n \sum_{l=1}^{n}\frac{(-1)^l}{l!}\frac{l}{2} f^{l-1}(q) f^{(1)}(q)\beta_{nl} \times \prod_{j=1}^{l}\left(\frac{\lambda(\lambda-1)}{2} - \frac{j(j-1)}{2}\right) \\ &= \sum_{n=1}^{\infty}(it)^n \sum_{l=1}^{n}\frac{(-1)^l}{l!}\frac{l}{2} f^{l-1}(q) f^{(1)}(q)\beta_{nl}\frac{\Gamma(\lambda+l)}{2^l\Gamma(\lambda-l)} \end{aligned} \tag{26}$$

where

$$\beta_{nl} = \frac{1}{2^{n-l}} \frac{(n-1)!}{(l-1)!} \frac{a_l^{(2n-2)}}{(2n-1)!} \tag{27}$$

To estimate the behavior of β_{nl} when $n \to \infty$, we probe the asymptotics of $a_l^{(2n-2)}$. Let us take $a_l^{(2n-2)}$ for sufficiently large n and express $a_l^{(2n-2)}$ with the help of (22) via the coefficients with smaller n and l so as to come to $a_1^{(0)} = 1$ at the end. The maximal contribution arises in this procedure when, at the beginning, n is diminished at fixed l by means of the first term on the right-hand side of (22), and then, when the equality $n = l - 1$ becomes valid, n and l start to decrease simultaneously by unity at every step by means of the second term in (22). For large n this gives the estimate

$$a_l^{(2n-2)} \sim 4^{n-l}\, l^{2(n-1)}\, (2l-1)!$$

Then β_{nl} behaves as

$$\beta_{nl} \sim 2^{n-l}\, l^{2(n-l)} \frac{(n-1)!}{(l-1)!} \frac{(2l-1)!}{(2n-1)!} \tag{28}$$

Now one can evaluate the asymptotics at $n \to \infty$ of the coefficients of the series (25), (26). Taking in (25), (26) in the sum over l the term with $l = n$, we obtain for noninteger λ that the coefficients in front of t^n grows in (25) as

$$\frac{f^n}{2^n} \frac{\Gamma(\lambda+n)}{n!\,\Gamma(\lambda-n)} \sim n!$$

and in (26) as

$$\frac{f^{n-1} f^{(1)}}{2^{n+1}} \frac{\Gamma(\lambda+n)}{(n-1)!\,\Gamma(\lambda-n)} \sim n!$$

So, for noninteger λ the series (25), (26), and, hence, (9) are asymptotic.

Let now λ be integer ($\lambda > 1$). Then in (25), (26) in the sums over l only the terms with $l < \lambda$ are different from zero, and, in fact, one should take instead of $\sum_{l=1}^{n}$ the sum $\sum_{l=1}^{\min\{n,\, \lambda-1\}}$. For $n \geq \lambda - 1$ the sum over l will always contain $\lambda - 1$ terms, and its dependence on n will be determined only by the dependence on n of the coefficients β_{nl}. And the dependence of the latter on n, as is clear from estimate (28), at fixed $l \leq \lambda - 1$ and at $n \to \infty$ is determined by the factor

$$\beta_{nl} \sim (2(\lambda-1)^2)^n \frac{(n-1)!}{(2n-1)!}$$

So the coefficients in front of t^n in (25), (26) behave at large n as

$$\frac{C^n\,(n-1)!}{(2n-1)!}$$

with any positive C, i.e., the series will be convergent at the circle of infinite range.

To obtain finally the function $F(t;\,q',\,q)$ it is necessary either to take the coefficients b_{n0}, b_{n1} from (25), (26) to calculate other b_{nk} using (23), or starting from B_0, B_1 calculate other functions $B_k(t,\,q)$ from the equation

$$B_{k+2} = \frac{2}{(k+1)(k+2)}\left(\frac{1}{i}\frac{\partial B_k}{\partial t} + \frac{k}{it}B_k + g\sum_{m=0}^{k}\frac{f^{(m)}}{m!}B_{k-m}\right) \tag{29}$$

which in an obvious way is derived from (6) after the substitution

$$F(t;\,q',\,q) = \sum_{k=0}^{\infty}\Delta q^k\,B_k\,(t,\,q) \tag{30}$$

and substitute them into (30).

In particular, for $\lambda = 2$ ($g = 1$) we have the potential $V(q) = -1/\cosh^2 q$, for which

$$B_0(t,\,q) = 1 - f(q)\sum_{n=1}^{\infty}\frac{(it)^n}{(2n-1)!!}$$
$$= 1 - f(q)\sqrt{\frac{\pi it}{2}}\,e^{it/2}\,\mathrm{erf}(\sqrt{it/2}) \tag{31}$$

$$B_1(t,\,q) = -\frac{1}{2}f^{(1)}\,(q)\sum_{n=1}^{\infty}\frac{(it)^n}{(2n-1)!!}$$
$$= -\frac{1}{2}f^{(1)}\,(q)\sqrt{\frac{\pi it}{2}}\,e^{it/2}\,\mathrm{erf}(\sqrt{it/2}) \tag{32}$$

With the help of (29) one is able to determine all coefficient functions B_k starting from (31), (32) and then substitute them into (30). In this manner the function F will be found.

We established that for integer λ the expansion (9) was convergent if $|\Delta q| < R(q)$ and the representation (4), (9) for the evolution operator kernel was not asymptotic. The function F is single-valued analytic in the entire complex plane of the variable t function and it has an essential singularity at the infinite ($t = \infty$) point.

The potential (17) is representative of a class of potentials studied in Osborn and Fujiwara (1983). It can be written as the Fourier transform

$$V(x) = \int_{-\infty}^{+\infty} e^{i\alpha x}\, d\mu(\alpha)$$

where

$$d\mu(\alpha) = -\frac{g\alpha\, d\alpha}{2\sinh(\pi\alpha/2)}$$

Osborn and Fujiwara (1983) showed that $|a_n| < n^{2n}/n! \sim n!$ when $n \to \infty$ for potentials of that class. This does not mean that the Schwinger–DeWitt expansion should be divergent in every case, because this is only bound from above, but not from below. So our result on the convergence of the expansion for the potential (17) for integer λ does not contradict the conclusions of Osborn and Fujiwara (1983).

4. POTENTIAL $V(q) = g/q^2$

We get another example of convergent series (9) by considering the potential

$$V(q) = \frac{\lambda(\lambda - 1)}{2} \frac{1}{q^2} \tag{33}$$

on the half-line $q > 0$. This potential is well studied and an analytic expression for its kernel is known. Our purpose is to show how the method described above can be applied to singular potentials.

Expansion (8) for the potential (33) has the finite convergence range $R(q) = q$, the finiteness of which is connected with the singularity of $V(q)$ at the point $q = 0$. The derivatives $V^{(k)}$ may be easily calculated

$$V^{(k)}(q) = (-1)^k \frac{\lambda(\lambda - 1)}{2} \frac{(k + 1)!}{q^{k+2}} \tag{34}$$

But for this potential an additional problem arises because of its singularity at the origin. To obtain the kernel that provides fulfilment of the boundary condition for the wave function $\psi(q)$ at $q = 0$[$\psi(q)$ should vanish at $q =$ 0] one needs to use an initial condition of more general form as compared with (2). Namely, in this case

$$\langle q', t = 0 | q, 0\rangle = \delta(q' - q) + A\delta(q' + q) \tag{35}$$

where the constant A is determined by the requirement that the kernel does not have a singularity at $q = 0$ and/or $q' = 0$ ($t \neq 0$). In correspondence with (35) and analogously to (4), the kernel may be represented as

$$\langle q', t|q, 0\rangle = \frac{1}{\sqrt{2\pi it}} \exp\left\{i\frac{(q'-q)^2}{2t}\right\} F^{(-)}(t; q', q)$$
$$+ A\frac{1}{\sqrt{2\pi it}} \exp\left\{i\frac{(q'+q)^2}{2t}\right\} F^{(+)}(t; q', q) \tag{36}$$

The equations for the functions $F^{(\pm)}$ are given by

$$i\frac{\partial F^{(\pm)}}{\partial t} = -\frac{1}{2}\frac{\partial^2 F^{(\pm)}}{\partial q'^2} + \frac{q' \pm q}{it}\frac{\partial F^{(\pm)}}{\partial q'} + V(q')F^{(\pm)} \tag{37}$$

and the initial conditions are

$$F^{(\pm)}(t = 0; q', q) = 1 \tag{38}$$

Directly in (37), (38), $q', q > 0$. But it is possible to consider the analytic continuation of $F^{(\pm)}$ into the complex plane q and adopt negative values of q. Then one may write

$$F^{(+)}(t; q', q) = F^{(-)}(t; q', -q) \tag{39}$$

and study only one function $F(t; q', q) = F^{(-)}(t; q', q)$, where q', q may be both positive and negative.

It is convenient to begin with positive q', q. In this case the technique described above can be used without any changes. From (10) with account of (34) we take

$$b_{nk} = \frac{1}{n+k}\left[\frac{(k+1)(k+2)}{2} b_{n-1k+2}\right.$$
$$\left. + \frac{\lambda(\lambda-1)}{2}\sum_{m=0}^{k}(-1)^{m+1}\frac{m+1}{q^{m+2}} b_{n-1,k-m}\right] \tag{40}$$

If we diminish n times the first index of b_{nk} by means of (40), then we get

$$b_{nk} = \frac{(-1)^{n+k}}{q^{2n+k}}\frac{(k+n-1)!}{n!(n-1)!k!}\prod_{j=1}^{n}\left(\frac{\lambda(\lambda-1)}{2} - \frac{j(j-1)}{2}\right)$$
$$= \frac{(-1)^{n+k}}{q^{2n+k}}\frac{(k+n-1)!}{n!(n-1)!k!}\frac{\Gamma(\lambda+n)}{2^n\Gamma(\lambda-n)} \tag{41}$$

It is obvious that for noninteger λ, $|b_{nk}| \sim n!$ when $n \to \infty$. So, for noninteger λ the expansion (9) for the potential (33) is divergent. But if λ is integer ($\lambda > 1$, because cases $\lambda = 0$, $\lambda = 1$ are trivial), then one can easy see from (41) that only the coefficients b_{nk} for $n < \lambda$ are different from zero, and in (9) the series in powers of t is really a polynomial of finite degree $\lambda - 1$.

Let us substitute (41) into (9) and make summation over k. Then we get finally

$$F(t; q', q) = 1 + \sum_{n=1}^{\infty} \left(\frac{-it}{2q'q} \right)^n \frac{\Gamma(\lambda + n)}{n!\Gamma(\lambda - n)} \tag{42}$$

Derivation of (42) is made in supposition that $|\Delta q| < |q|$. If this condition is not satisfied, then calculations should be made with the expansion about the point q' in powers of $q - q'$. But because F is symmetric in q', q, then it is clear that the answer in this case would be the same as in (42). Note that the representation (42) for F does not suppose that $q', q > 0$. One can put q' and/or q negative and, hence, (42) gives expressions both for $F^{(-)}$ and for $F^{(+)}$.

Expansion (42) has a singularity at $q = 0$ ($q' = 0$). For noninteger λ this expression is asymptotic, so it cannot be applied to analyze the behavior of F at $q', q \to 0$. It is correct only for sufficiently small values of variable $t/q'q$. But for integer λ, (42) becomes a finite series because in this case $1/\Gamma(\lambda - n) = 0$ for $n \geq \lambda$ and the sum over n should be made only to $\lambda - 1$, but not to infinity. One can state, thus, that $F^{(\pm)}$ is really singular at the point $q = 0$ (or $q' = 0$). This feature, nevertheless, is not so dangerous, because only the kernel (36) should be finite at $q = 0$ ($q' = 0$).

After substitution of (42) into (36) we get

$$\begin{aligned} \langle q', t | q, 0 \rangle = \frac{1}{\sqrt{2\pi i t}} \exp\left\{ i \frac{q'^2 + q^2}{2t} \right\} \Bigg\{ & e^{-iq'q/t} \sum_{n=0}^{\lambda-1} \left(\frac{-it}{2q'q} \right)^n \frac{\Gamma(\lambda + n)}{n!\Gamma(\lambda - n)} \\ & + A e^{iq'q/t} \sum_{n=0}^{\lambda-1} \left(\frac{it}{2q'q} \right)^n \frac{\Gamma(\lambda + n)}{n!\Gamma(\lambda - n)} \Bigg\} \end{aligned} \tag{43}$$

Expanding $\exp\{\pm iq'q/t\}$ into series in $q'q/t$ and considering terms with a singularity in variable $q'q$, we can see that if $A = e^{-i\pi\lambda}$, then all these terms will be canceled and the kernel will be equal to zero when $q'q = 0$ (this will be zero of order λ). So, the initial condition (35) with $A = e^{-i\pi\lambda}$ provides fulfillment of the boundary condition for the kernel at the origin.

Finally the kernel may be represented as

$$\begin{aligned} \langle q', t | q, 0 \rangle = & \frac{1}{\sqrt{2\pi i t}} \exp\left\{ i \frac{(q' - q)^2}{2t} \right\} \sum_{n=0}^{\infty} \left(\frac{-it}{2q'q} \right)^n \frac{\Gamma(\lambda + n)}{n!\Gamma(\lambda - n)} \\ & + e^{-i\pi\lambda} \frac{1}{\sqrt{2\pi i t}} \exp\left\{ i \frac{(q' + q)^2}{2t} \right\} \sum_{n=0}^{\infty} \left(\frac{it}{2q'q} \right)^n \frac{\Gamma(\lambda + n)}{n!\Gamma(\lambda - n)} \end{aligned} \tag{44}$$

For integer λ, sums are made only to $\lambda - 1$ and this expansion is finite; for noninteger λ it is asymptotic.

Naturally, this result exactly coincides with well-known representation

$$
\begin{aligned}
\langle q', t|q, 0\rangle
&= e^{-i(\pi/2)(\lambda-1/2)} \frac{\sqrt{q'q}}{it} \exp\left\{i\frac{q'^2+q^2}{2t}\right\} J_{\lambda-1/2}\left(\frac{q'q}{t}\right)\\
&= \frac{1}{\sqrt{2\pi it}} \exp\left\{i\frac{(q'-q)^2}{2t}\right\} \sqrt{\frac{\pi q'q}{2it}}\, e^{-i(\pi/2)(\lambda-1/2)}\, e^{iq'q/t}\, H^{(2)}_{\lambda-1/2}\left(\frac{q'q}{t}\right)\\
&\quad \times + e^{-i\pi\lambda} \frac{1}{\sqrt{2\pi it}} \exp\left\{i\frac{(q'+q)^2}{2t}\right\}\\
&\quad \times \sqrt{\frac{\pi q'q}{2it}}\, e^{i(\pi/2)(\lambda+1/2)}\, e^{-iq'q/t}\, H^{(1)}_{\lambda-1/2}\left(\frac{q'q}{t}\right)
\end{aligned}
\tag{45}
$$

which may be derived directly by reducing the Schrodinger equation to the equation for cylindrical functions (here J_ν is the Bessel function, $H_\nu^{(1,2)}$ are Hankel functions of the first and second kinds). Strictly speaking, the validity of (44) for noninteger λ follows only from (45), but not from previous considerations.

Note that the essential feature of the representation (44) for the kernel is the following: the sums over index n are divergent if λ is noninteger and are finite if λ is integer. The kernel is well defined in both cases and the apparent expression (45) allows us to study its behavior in different variables: t, q', q, or λ. But if we do not know the exact solution, as usually occurs in more complicated problems with other potentials, and if we have only an asymptotic expansion of the form (36), then we will have many problems for noninteger λ and much fewer problems for integer λ.

For example, for the potential (33) we cannot study from the asymptotic expansion (44) the behavior of the kernel at $q \to 0$. In particular, if we did not have the exact expression (45), but only the asymptotic one (44), we would not know that the kernel has a zero of order λ at $q'q \to 0$.

From the other side, if we try to get the expansion in powers of the coupling constant $g = \lambda(\lambda - 1)/2$ for F starting from (42), we would get after some transformations

$$
F(t; q', q) = 1 + \sum_{k=1}^{\infty} g^k \sum_{n=k}^{\infty} \left(\frac{-it}{q'q}\right)^n \frac{C_{nk}}{n!}
\tag{46}
$$

where C_{nk} are the coefficients of the polynomial

$$
\prod_{j=1}^{n}\left(g - \frac{j(j-1)}{2}\right) = \sum_{k=1}^{n} g^k C_{nk}
\tag{47}
$$

Equation (46) is a series of conventional perturbation theory. This series is not simply divergent, but its coefficients are divergent, too. Consider the contribution of the first order in g:

$$F^{(1)}(t; q', q) = g \sum_{n=1}^{\infty} \left(\frac{-it}{q'q}\right)^n \frac{C_{n1}}{n!}$$

From (47) one can easy derive

$$C_{n1} = (-1)^{n-1} \frac{n!(n-1)!}{2^{n-1}}$$

Hence,

$$F^{(1)}(t; q', q) = -2g \sum_{n=1}^{\infty} \left(\frac{-it}{2q'q}\right)^n (n-1)! \tag{48}$$

and the coefficient in front of g is a divergent series. Now we see that without knowledge of the exact solution for noninteger λ we cannot build even conventional perturbation theory for F. Nevertheless, for integer λ the Schwinger–DeWitt expansion is convergent and it can be used for further applications.

5. OTHER EXAMPLES OF POTENTIALS

The calculations made in Sections 3 and 4 may be easily repeated for some similar potentials which are often used in one-dimensional many-body problems (Olshanetsky and Perelomov 1983; Calogero *et al.*, 1975; Sutherland, 1971, 1972). These are the potentials

$$V(q) = \frac{\lambda(\lambda-1)}{2} \frac{1}{\sinh^2 q} \tag{49}$$

and

$$V(q) = \frac{\lambda(\lambda-1)}{2} \frac{1}{\sin^2 q} \tag{50}$$

To prove convergence of the series for $F^{(-)}(t; q', q)$ $(q', q > 0)$ it is enough to make a little modification of the considerations of Section 3.

For the potential (49) denote

$$f(q) = \frac{1}{\sinh^2 q} \tag{51}$$

and notice that

$$(f^{(1)})^2 = 4f^3 + 4f^2$$

i.e., it exactly coincides with the corresponding expression for the function $f(q)$ defined by (18) in Section 3. So all relations for the derivatives of f obtained there and hence expressions for b_{nk}, B_k, and F (now $F^{(-)}$) hold in this case. There exist only two differences: the function f is defined now by (51), but not by (18), and the convergence range of the expansion (8) is $R(q) = (\pi^2 + q^2)^{1/2}$.

Hence, convergence of (5) for the potential (49) when $q', q > 0$ occurs for integer λ. The function $F^{(-)}$ is a single-valued and analytic function in the entire complex plane of the variable t for all $q', q > 0$.

The potential (50) in the region $0 < q < \pi$ also can be considered in a similar way. Denote

$$f(q) = \frac{1}{\sin^2 q} \tag{52}$$

and take into account that

$$(f^{(1)})^2 = 4f^3 - 4f^2$$

This expression differs from analogous ones for the potentials (17), (49) only by the sign of the second term. So we are able to reconstruct the expressions from Section 3 with only small changes: in (25), (26) there will appear an additional multiplier $(-1)^{n+1}$, and the function f will be defined by (52).

The conclusion about convergence of the expansion (5) for $F^{(-)}$ in the region $0 < q', q < \pi$ when λ is integer holds for the potential (50). But both these potentials are singular at $q = 0$. So one is to consider the initial condition (35) and the additional function $F^{(+)}$ $(t; q', q)$ as in Section 4. It is enough to continue $F^{(-)}$ into the region $q < 0$ and use (38). Expressions obtained in Section 3 do not allow us to make any conclusions about the behavior of F for $q < 0$.

Let us consider another representation for $a_n(q', q)$. The potentials (49), (50) may be written as follows:

$$V(q) = g\left(\frac{1}{q^2} + \sum_{k=0}^{\infty} s_k q^k\right) \tag{53}$$

where s_k are known coefficients, $g = \lambda(\lambda - 1)/2$. The coefficient functions $a_n\ (q', q)$ have the form of Loran series with a finite number of pole terms

$$a_n(q', q) = \sum_{k=-n}^{\infty} \sum_{l=-n}^{\infty} q'^k\, q^l\, a^n_{kl} \tag{54}$$

Substitution of (5), (53), and (54) into (36) gives us algebraic recurrence relations for d^n_{kl},

$$(n + k)d^n_{kl} \pm (k + 1)d^n_{k+1,l}$$
$$= \left(\frac{(k + 1)(k + 2)}{2} - g\right) d^{n-1}_{k+2,l} - g \sum_{m=k+n-1}^{\infty} s_m d^{n-1}_{k-m,l} \qquad (55)$$

If $k + n - 1 < 0$, then the sum in the last term of (55) should be equated to zero. One can find solutions of equation (55) and be convinced of the validity of the representation (54). It is obvious, that the terms in a_n most singular at q', $q = 0$ do not depend on s_m and they exactly coincide with the ones for the potential (33).

We know that $F^{(-)}$ $(q', q > 0)$ for the considered potentials is represented by convergent series of type (9). Then there exist representations of type (5), (54) and the Loran series (54) is convergent at the polycircle of finite radii pierced at zero. For this series the sign of q has no meaning. One may consider both $q > 0$ and $q < 0$. Convergence will take place in any case.

One can state that convergence of expansion (5) for $F^{(+)}$ follows from its convergence for $F^{(-)}$. The latter was proved earlier. So we established that for the potentials (49), (50) the Schwinger–DeWitt expansion (36) is convergent for integer λ. Singularities of $F^{(\pm)}$ at q', $q = 0$ cancel each other in the combination (36) if $A = e^{-i\pi\lambda}$ as in the case of the potential (33).

6. CONCLUSION

Usually the Schwinger–DeWitt expansion is used asymptotically. Its general property an increase of the coefficients $a_n(q', q)$ as $n!$ for $n \to \infty$ [or as $\Gamma(n(L - 2)/(L + 2))$ if the potential is polynomial of order L) (Osborn and Fujiwara, 1983; Slobodenyuk, 1995). Such growth always takes place when no cancellations of different contributions to a_n occur. This is so for most potentials. But there exist some potentials for which cancellations occur. Examples of such potentials were considered in present paper. We proved the convergence of the Schwinger–DeWitt expansion for them when the constant λ is integer and divergence when λ is noninteger.

Besides the potentials mentioned above, one more example is known (Slobodenyuk, 1996), which has the property of convergence of the expansion (9):

$$V(q) = a^2 q^2 + \frac{\lambda(\lambda - 1)}{2} \frac{1}{q^2} \qquad (56)$$

The expansion (5) converges for this when λ is integer. But the convergence range is finite, contrary to the examples of this paper. It is natural because the expansion for the harmonic oscillator $V(q) = a^2q^2$ has a finite convergence range. And this is so for the perturbed oscillator (56), too.

So, we have discovered the existence of a class of nontrivial potentials in quantum mechanics for which the Schwinger–DeWitt expansion is convergent and for which the initial condition for the evolution problem is fulfilled in a rigorous (analytic) sense, but not only asymptotically (when F is not analytic at $t = 0$ and its value at the origin is determined from the condition of continuity).

The potentials belonging to this class have at least two remarkable features: (1) the Schwinger–DeWitt expansion is convergent for them, hence, other expansions which may be derived from it are convergent, too, and many problems connected with the divergence of the expansions are absent for such potentials, and (2) the potentials of this class have discrete coupling constants, which corresponds to the discreteness of charge in nature.

This is why the potentials from this class are well studied. But research using quantum mechanical models is only preparation for practical use relating to physical phenomenon. One can hope to construct a fundamental theory of elementary particles as a result of applying this approach to quantum field theory. It is possible to introduce interaction in the field theory analogous to the quantum mechanical potentials studied in this article. One such quantum field model is under consideration and will be described in a subsequent paper. The model conserves the essential features discussed above.

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